The structure of the normalizers of maximal tori in groups of Lie type

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Classification of Finite Simple Groups. Why?

A group is said to be *simple* if it has no non-trivial proper normal subgroups.

The idea of a simple group is easy: just as a prime number is a number which cannot be factorised into two smaller numbers, so a group is *simple* if it cannot be broken up into two smaller groups.

The importance of simple groups follows from the *Jordan-Hölder Theorem*, proved around 1889. It tells us that just as all molecules are built from atoms, and all positive integers are built from prime numbers, so all finite groups are built from finite simple groups. Once you understand the finite simple groups, you understand a lot about all finite groups.

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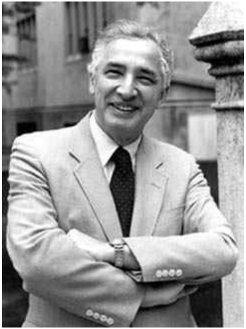
Classification of Finite Simple Groups. History



Theorem (Walter Feit and John Griggs Thompson, 1963)

Every finite group of odd order is solvable. In particular, every finite nonabelian simple group contains an element of order 2.

255 pages of proof – entire issue of «Pacific Journal of Mathematics». J.G. Thompson was awarded the Fields Medal in 1970 (Noble Prize in mathematics).

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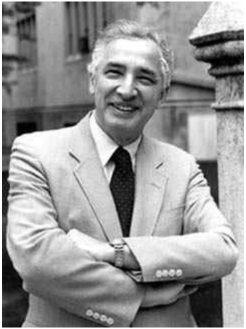

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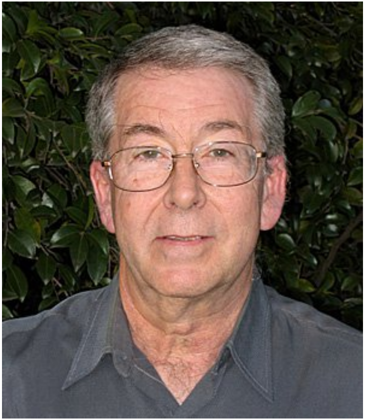
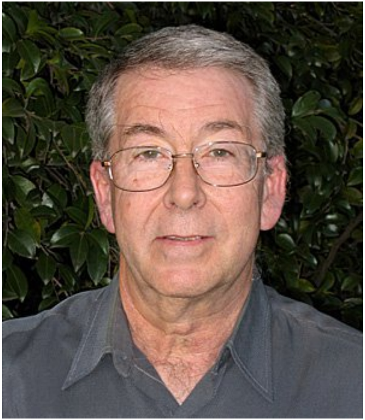
- 1983 – Gorenstein announces that the classification is complete! He called it «Thirty Years War», for the Classification battles were fought mostly in the decades 1950s–1980s.

Classification of Finite Simple Groups. History



- 1997 – Michael Aschbacher announces that it is not!!

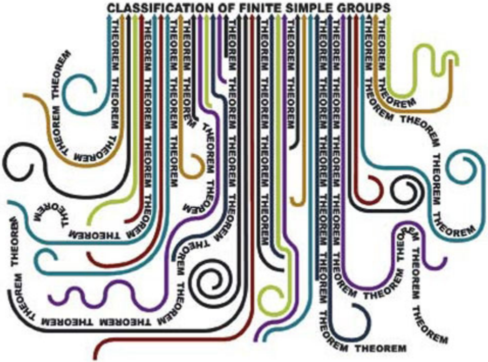
Classification of Finite Simple Groups. History



- 1997 – Michael Aschbacher announces that it is not!!
- 2004 – Michael Aschbacher and Stephen Smith fill in the gap (across 1221 pages)

Classification of Finite Simple Groups

- The first proof took about 10.000 pages, spread across 500 or so journal articles, by over 100 different authors from around the world; it must be counted the longest proof in history.



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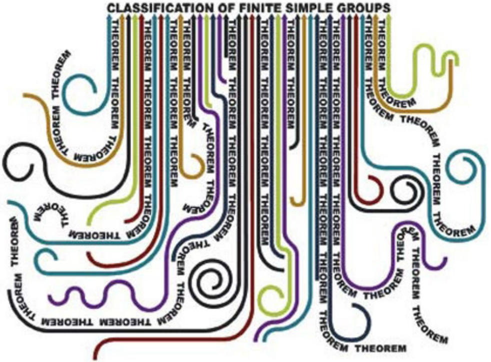

Victor Mazurov
(my supervisor)

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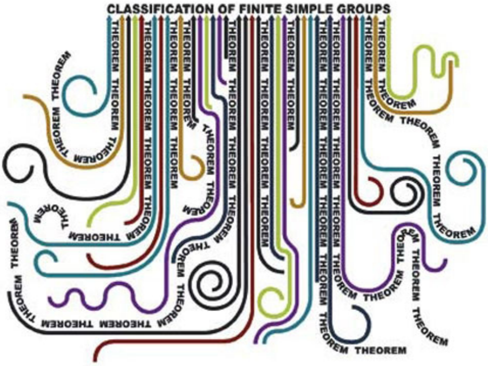

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- Work began almost immediately on a «second generation» proof. This is an ongoing project...

Theorem (Classification of Finite Simple Groups)

Every finite simple group is isomorphic to one of the following groups:

- $\mathbb{Z}_p$ for p prime
- Alternating groups $\mathrm{Alt}(n)$ for $n \geqslant 5$
- 16 infinite families of groups of Lie type
- One of the 26 sporadic simple groups

Groups of Lie type can be uniformly described using Dynkin diagrams.

What are the sporadic simple groups?
They don't fall into any other families. The fact that these groups exist is one reason why the classification is so difficult.

The largest of these groups is called the «Monster». It was constructed by Robert Griess in 1982 and has 808017424794512875886459904961710757005754368000000000 elements!

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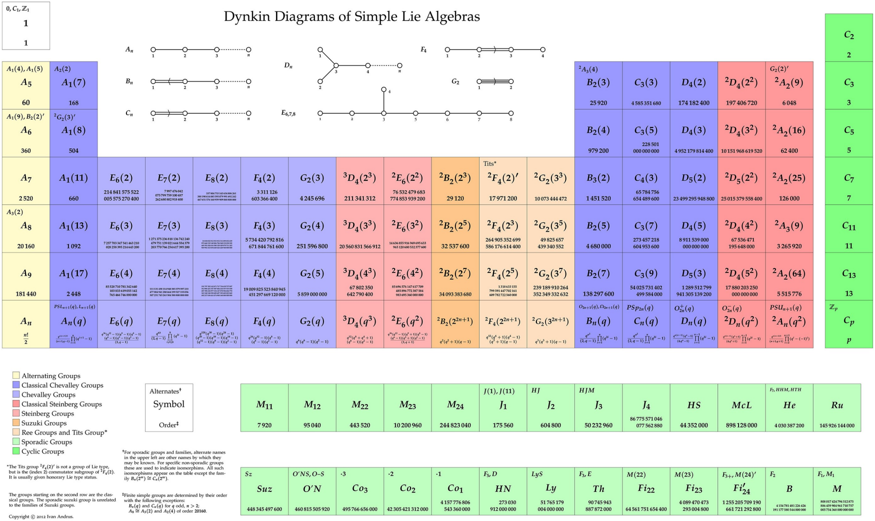
Classification Table



The extension problem

A second major theme in finite group theory is the *extension problem* for finite groups.
As discussed, the simple groups act for group theory like prime numbers do for number theory. However, there is a problem:

	Number Theory	Group Theory
Classify all «primes»	HARD	HARD BUT DONE
Construct elements from their «prime factors»	VERY EASY	HARD AND NOT DONE

Numbers have unique factorisation, so it is very easy to construct the unique number with a given «prime factors». For groups this isn't anything like as straightforward.

Example of different constructions

Example

Let $G = \langle (1,2)(3,4)(5,6)(7,8), (2,4,8,9)(3,7,6,10) \rangle$. In fact $G \simeq \mathrm{Alt}(6)$, and so $|G| = 360$. There are various ways to add an element to G to give us a group of order 720.

- Adding $s_1 := (1,3,7,4,2,8)(5,10,9)$ gives $\mathrm{Sym}(6)$.
- Adding $s_2 := (1,8,4,6,2,7,3,5)$ gives $\mathrm{PGL}(2,9)$.
- Adding $s_1 s_2$ gives M_{10}, the stabiliser of a single point in the sporadic simple group M_{11}.

These all have the same simple factors, namely $\mathrm{Alt}(6)$ and $\mathbb{Z}_2$, but none of these three groups are isomorphic.

The extension problem

A complete answer to such question would give a classification of all finite groups. But in full generality it seems now inaccessibly hard.

Given two abstract groups H and N, we can ask for a classification of all groups G such that G has a normal subgroup N and $G/N \simeq H$. Here G is called an *extension* of H by N, and finding this classification is known as the *extension problem*.

The groups with simple factors $\mathrm{Alt}(6)$ and $\mathbb{Z}_2$ are:

- *A direct product* $\mathrm{Alt}(6) \times \mathbb{Z}_2$.
- *A semidirect product* (*a split extension*) $\mathrm{Alt}(6) \rtimes \mathbb{Z}_2$. All three of the groups described in the Example fall into this category (and these are the only such ones in this case)
- *A nonsplit extension* – this is a group G such that H is contained in the center of G, and $G/H \simeq K$. There is a perfect central extension of $\mathrm{Alt}(6)$ by $\mathbb{Z}_2$, usually denoted $2^{\cdot}\,\mathrm{Alt}(6)$. Here this is $\mathrm{SL}(2,9)$.

This is all possible groups with simple factors $\mathrm{Alt}(6)$ and $\mathbb{Z}_2$.

What about my research?

As we can see from Example the extensions can be split and nonsplit. It is a hard question to solve is the extension split or not?

Jacques Tits, awarded the Abel Prize in 2008 (with J.G. Thompson)

- J. Tits, *Normalisateurs de tores I. Groupes de Coxeter Étendus*, J. Algebra, **4** (1966), 96–116.
- The splitting problem for the normalizer of a maximal torus in groups of Lie type was formulated in this article.
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Definition

$T \lhd N$. A subgroup H is a *complement* for T in N, if $TH = N$ and $T \cap H = 1$, and if T has a complement in N, we say that N *splits* over T.

Example

$\mathbb{Z}_2 \lhd \mathbb{Z}_4$. $\mathbb{Z}_2$ has no complement in $\mathbb{Z}_4$.

Example

$\mathrm{A}_5 \lhd \mathrm{S}_5$. $K = \langle (1,2) \rangle$ is a complement for A_5 in S_5.

Example

$\overline{G} = \mathrm{SL}_2(\overline{\mathbb{F}}_p)$, where p is odd.

$\overline{T} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \overline{\mathbb{F}}_p^* \right\}$ is a maximal torus of $\overline{G}$.

$N_{\overline{G}}(\overline{T})$ is the group of all monomial matrices of $\overline{G}$ and $N_{\overline{G}}(\overline{T})/\overline{T} \simeq \mathrm{S}_2$.

$$\text{Does } N_{\overline{G}}(\overline{T}) \text{ split over } \overline{T}?$$

The group $\overline{G}$ contains only one element of order two:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and this element lies in $\overline{T}$.

Hence, $N_{\overline{G}}(\overline{T})$ does not split over $\overline{T}$.

Example

$\overline{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is the general linear group over $\overline{\mathbb{F}}_p$.
$\overline{T} = D_n(\overline{\mathbb{F}}_p)$ is a maximal torus of $\overline{G}$.
$N_{\overline{G}}(\overline{T})$ is the group of all monomial matrices of $\overline{G}$ and $N_{\overline{G}}(\overline{T})/\overline{T} \simeq \mathrm{S}_n$.
There exists a canonical embedding of S_n into the group of all monomial matrices of $\overline{G}$.
If $\overline{H}$ is an image of S_n under this embedding, then $\overline{H}$ is a complement for $\overline{T}$ in $N_{\overline{G}}(\overline{T})$.

Since the center $\mathrm{Z}(\overline{G})$ of $\overline{G}$ is contained in $\overline{T}$, then a maximal torus of $\mathrm{PGL}_n(\overline{\mathbb{F}}_p)$ also has a complement in their normalizer.

Moreover, $\mathrm{PGL}_n(\overline{\mathbb{F}}_p) \simeq \mathrm{PSL}_n(\overline{\mathbb{F}}_p)$ and the same is true for $\mathrm{PSL}_n(\overline{\mathbb{F}}_p)$.

Problems

Let $\overline{G}$ be a simple connected linear algebraic group over the algebraic closure $\overline{\mathbb{F}}_p$ of a finite field of positive characteristic p. Let σ be a Steinberg endomorphism and $\overline{T}$ a maximal σ-invariant torus of $\overline{G}$. It's well known that all the maximal tori are conjugated in $\overline{G}$ and the quotient $N_{\overline{G}}(\overline{T})/\overline{T}$ is isomorphic to the Weyl group W of $\overline{G}$.

Problem 1 (J. Tits)

Describe the groups $\overline{G}$ in which $N_{\overline{G}}(\overline{T})$ splits over $\overline{T}$.

J.Tits «Normalisateurs de tores I. Groupes de Coxeter Étendus» Journal of Algebra, 1966, V.4, 96–116.

Problems

A similar problem arises in finite groups G of Lie type.

Let G be a finite group of Lie type, that is $O^{p'}(\overline{G}_\sigma) \leqslant G \leqslant \overline{G}_\sigma$. Let $T = \overline{T} \cap G$ be a maximal torus of G and $N(G,T) = N_{\overline{G}}(\overline{T}) \cap G$ be an algebraic normalizer of T.

Problem 2

Describe the groups G and their maximal tori T in which $N(G,T)$ splits over T.

Problem 1 (J. Tits)

Problem 1 for simple Lie groups was solved in

M. Curtis, A. Wiederhold, B. Williams, «Normalizers of maximal tori» Springer, Berlin, 1974, Lecture Notes in Math., V. 418, 31–47.

An answer to Problem 1 was independently obtained in the paper

A. G., On the splitting of the normalizer of a maximal torus in the exceptional linear algebraic groups, Izvestiya: Mathematics, 2017, 81:2, 269–285.

and in the paper

J. Adams and X. He, «Lifting of elements of Weyl groups», J. Algebra 485 (2017), 142–165.

Algebraic groups

The answer to Problem 1 is in the following table:

Group	Conditions for the existence of a complement
$\mathrm{SL}_n(\overline{\mathbb{F}}_p)$	$p = 2$ or n is odd
$\mathrm{PSL}_n(\overline{\mathbb{F}}_p)$	—
$\mathrm{Sp}_{2n}(\overline{\mathbb{F}}_p)$	$p = 2$
$\mathrm{PSp}_{2n}(\overline{\mathbb{F}}_p)$	$p = 2$ or $n \leqslant 2$
$\mathrm{SO}_{2n+1}(\overline{\mathbb{F}}_p)$	—
$\mathrm{SO}_{2n}(\overline{\mathbb{F}}_p)$	—
$\mathrm{PSO}_{2n}(\overline{\mathbb{F}}_p)$	—
$G_2(\overline{\mathbb{F}}_p)$	—
$F_4(\overline{\mathbb{F}}_p)$	$p = 2$
$E_6(\overline{\mathbb{F}}_p)$	$p = 2$
$E_7(\overline{\mathbb{F}}_p)$	$p = 2$
$E_8(\overline{\mathbb{F}}_p)$	$p = 2$

Problem 2

Let $O^{p'}(\overline{G}_\sigma) \leqslant G \leqslant \overline{G}_\sigma$ be a finite group of Lie type. Two maximal tori in G are not necessary conjugate in G. Let W be a Weyl group of $\overline{G}$, π a natural homomorphism from $\overline{N} = N_{\overline{G}}(\overline{T})$ into W.

Two elements w_1, w_2 are called σ-conjugate if $w_1 = w^{-1}w_2w^\sigma$ for some element w of W.

Proposition

There is a bijection between the G-classes of σ-stable maximal tori of $\overline{G}$ and the σ-conjugacy classes of W.

Problem 2

Define $C_{W,\sigma}(w) = \{x \in W | x^{-1}wx^{\sigma} = w\}$.

Proposition

Let $g^{\sigma}g^{-1} \in \overline{N}$ and $\pi(g^{\sigma}g^{-1}) = w$. Then

$$(N_{\overline{G}}(\overline{T}^{g}))_{\sigma}/(\overline{T}^{g})_{\sigma} \simeq C_{W,\sigma}(w).$$

In this case we say that $T = (\overline{T}^{g})_{\sigma}$ corresponds to w.

Linear groups

$W(A_{n-1}) \simeq \mathrm{S}_n$
σ-conjugacy classes in $W \longleftrightarrow$
ordinary conjugacy classes in $\mathrm{S}_n \longleftrightarrow$
partition $\{n_1, \ldots, n_m\}$ of n

What is the structure of $C_{\mathrm{S}_n}(w)$?

Example

$$w = \underbrace{\underbrace{(123)}_{\mathbb{Z}_3}\underbrace{(456)}_{\mathbb{Z}_3}}_{\mathrm{S}_2}\underbrace{(78)}_{\mathbb{Z}_2}$$

$$C_{\mathrm{S}_n}(w) = (\langle(123)\rangle \times \langle(456)\rangle) \rtimes \langle(14)(25)(36)\rangle \times \langle(78)\rangle \simeq$$

$$(\mathbb{Z}_3 \wr \mathrm{S}_2) \times \mathbb{Z}_2$$

Linear groups

Let $\{n_1, \ldots, n_m\}$ be a partition of n. We assume that

$$n_1 = \ldots = n_{l_1} < \ldots < n_{l_1+\ldots+l_{r-1}+1} = \ldots = n_{l_1+\ldots+l_r}$$

Define

$$a_1 = n_{l_1} l_1, \ldots, a_r = n_{l_1+\ldots+l_r} l_r$$

$$w = \overbrace{\underbrace{(\ldots)}_{n_1}\underbrace{(\ldots)}_{n_1}\ldots\underbrace{(\ldots)}_{n_1}}^{a_1}\ldots\overbrace{\underbrace{(\ldots)}_{n_m}\underbrace{(\ldots)}_{n_m}\ldots\underbrace{(\ldots)}_{n_m}}^{a_r}$$

$$C_{\mathrm{S}_n}(w) = (\mathbb{Z}_{n_1} \wr \mathrm{S}_{l_1}) \times \ldots \times (\mathbb{Z}_{n_m} \wr \mathrm{S}_{l_r})$$

Theorem (A. G.)

Let T be a maximal torus of $G = \mathrm{SL}_n(q)$ corresponding to an element of the Weyl group with cycle-type $(n_1)(n_2)\ldots(n_m)$. Then T has a complement in $N \Leftrightarrow q$ is even or a_i is odd for some $1 \leqslant i \leqslant r$.

Linear groups

Theorem (A. G.)

Let T be a maximal torus of $G = \mathrm{SL}_n(q)$ corresponding to an element of the Weyl group with cycle-type $(n_1)(n_2)\dots(n_m)$. Then T has a complement in N $\Leftrightarrow$ q is even or a_i is odd for some $1 \leqslant i \leqslant r$.

An element $w = e = (1)(2)\dots(n) \longleftrightarrow$ cycle-type $(1)(1)\dots(1)$,

$$C_W(e) = W,$$

$$a_1 = n_1 \cdot l_1 = 1 \cdot n = n.$$

T has a complement in N $\Leftrightarrow$ q is even or n is odd.

Group	Conditions for the existence of a complement
$\mathrm{SL}_n(\overline{\mathbb{F}}_p)$	$p = 2$ or n is odd

Linear groups

Let T be a maximal torus of $G = \mathrm{SL}_n(q)$ with the cycle-type $(n_1)(n_2)\dots(n_m)$. Let $\widetilde{T}$ and $\widetilde{N}$ be the images of T and N in $\widetilde{G} = \mathrm{PSL}_n(q)$. Then $\widetilde{T}$ has a complement in $\widetilde{N}$ $\Leftrightarrow$ one of the following holds:

(1) q is even;

(2) a_i is odd for some $1 \leqslant i \leqslant r$;

(3) $(n)_2 < (q-1)_2$;

(4) $m = 4$, n_1, n_2, n_3, n_4 are odd;

(5) $m = 3$, $n_1 = n_2$ is odd, n_3 is even, $(n_3)_2 > 2$, $(n)_2 \leqslant (q-1)_2$;

(6) $m = 3$, $n_1 = n_2$ is odd, n_3 is even, $(n_3)_2 = 2$, $(q-1)_2 \neq (n)_2$;

(7) $m = 2$, n_1, n_2 are odd;

(8) $m = 2$, n_1, n_2 are even, $n_1 \neq n_2$, $(n)_2 < d(q-1)_2$, where $d = \gcd\{(\frac{n_1}{2})_2, (\frac{n_2}{2})_2, (q-1)_2\}$;

(9) $m = 2$, n_1, n_2 are even, $n_1 \neq n_2$, $(n_1)_2 = (n_2)_2 \leqslant (q-1)_2$, $(q-1)_2(n_1)_2 \leqslant (n)_2$;

(10) $m = 2$, $n_1 = n_2$ is even, $(n_1)_2 > 2$, $(n)_2 \leqslant (q-1)_2$;

(11) $m = 2$, $n_1 = n_2$ is even, $(n_1)_2 = 2$, $(n)_2 \neq (q-1)_2$;

(12) $m = 1$.

Classical groups

Theorem (A. G.)

Let T be a maximal torus of G corresponding to an element of the Weyl group with cycle-type $(\overline{n_1})\dots(\overline{n_k})(n_{k+1})\dots(n_m)$. Then T has a complement in N $\Leftrightarrow$ one of the following holds:

G	Condition for the existence of a complement
$\mathrm{Sp}_{2n}(q)$	q is even
$\Omega_{2n+1}(q)$	$q \not\equiv 3 \pmod 4$
	a_i is odd for some $1 \leqslant i \leqslant r$
	$k = m$, n_i is even for every $1 \leqslant i \leqslant m$
$\Omega^{\varepsilon}_{2n}(q)$	$q \not\equiv 3 \pmod 4$
	a_i is odd for some $1 \leqslant i \leqslant r$
	$k = m$, n_i is even for every $1 \leqslant i \leqslant m$
$\mathrm{SL}^{\varepsilon}_n(q)$	q is even
	a_i is odd for some $1 \leqslant i \leqslant r$

Classical groups

Theorem (A. G.)

Let T be a maximal torus of G corresponding to an element of the Weyl group with cycle-type $(\overline{n_1})\dots(\overline{n_k})(n_{k+1})\dots(n_m)$, $m > 4$. Then T has a complement in N $\Leftrightarrow$ one of the following holds:

G	Condition for the existence of a complement
$\mathrm{PSp}_{2n}(q)$	q is even
$\Omega_{2n+1}(q)$	$q \not\equiv 3 \pmod 4$
	a_i is odd for some $1 \leqslant i \leqslant r$
	$k = m$, n_i is even for every $1 \leqslant i \leqslant m$
$\mathrm{P}\Omega^{\varepsilon}_{2n}(q)$	$q \not\equiv 3 \pmod 4$
	a_i is odd for some $1 \leqslant i \leqslant r$
	$k = m$, n_i is even for every $1 \leqslant i \leqslant m$
$\mathrm{PSL}^{\varepsilon}_n(q)$	q is even
	a_i is odd for some $1 \leqslant i \leqslant r$
	$(n)_2 < (\varepsilon q - 1)_2$

Exceptional groups $F_4(q), E_6(q), E_7(q), E_8(q)$, q is odd

$F_4(q)$: T does not have a complement in $N \Leftrightarrow |w|$ divides 4

$E_7(q)$: T does not have a complement in $N \Leftrightarrow |w|$ divides 4

$E_6(q)$: T does not have a complement in $N \Leftrightarrow |w|$ divides 4, except

№	w	$\lvert w\rvert$	$\lvert C_W(w)\rvert$	$C_W(w)$	Complement
14	$w_3w_2w_4w_{14}$	4	96	$\mathrm{SL}_2(3):\mathbb{Z}_4$	$q \not\equiv 3 \pmod 4$

$E_8(q)$: T does not have a complement in $N \Leftarrow |w|$ divides 4

Exceptional groups of a small rank

Theorem (A. G. and Alexey Staroletov)

Let $G \in \{^3D_4(q), G_2(q), ^2G_2(q)\}$ and T be a maximal torus of G. Then T has a complement in $N(G,T)$.

Main theorem (A. G. and Alexey Staroletov)

Let G be a finite simple group of Lie type and let T be a maximal torus of G. Assume that the algebraic normalizer $N(G,T)$ splits over T. Then the pair (G,T) is described.

Thank you for your attention!

Novosibirsk, Academgorodok.